BIRZEIT UNIVERSITY

B321

Name:________________ MATH [redacted] Summer Semester2016

Number:________________ 1st. Exam

Q#1 (20 points) Evaluate the following integrals

a- $\int_2^\infty \frac{dx}{x^2 - x}$

$$\frac{1}{x^2-x} = \frac{1}{x(x-1)} = \frac{A}{x-1} + \frac{B}{x}$$

$$1 = Ax + B(x-1)$$

$$x=0, \quad B=-1$$

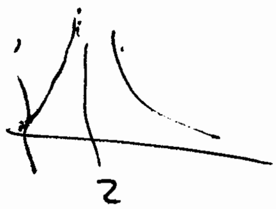
$$x=1, \quad A=1$$

$$\int_2^\infty \frac{dx}{x^2-x} = \lim_{b\to\infty}\int_2^b \frac{1}{x^2-x} = \lim_{b\to\infty}\int_2^b \left(\frac{1}{x-1} - \frac{1}{x}\right)dx \quad (3)$$

$$= \lim_{b\to\infty}\left(\ln|x-1| - \ln x \Big|_2^b\right] = \lim_{b\to\infty}\left[\ln\left(\frac{x-1}{x}\right)\Big|_2^b\right] \quad (3)$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{b-1}{b}\right) - \ln\left(\frac{1}{2}\right)\right] = \ln 1 - \ln\frac{1}{2} = \ln 2 \quad (4)$$

b- $\int_0^2 \frac{dx}{(x-2)^{\frac{2}{5}}}$



$$= \lim_{b\to 2^-}\int_0^b \frac{dx}{(x-2)^{2/5}} \quad (3)$$

$$= \lim_{b\to 2^-}\left[\frac{5}{3}(x-2)^{3/5}\Big|_0^b\right] \quad (3)$$

$$= \lim_{b\to 2^-}\left[\frac{5}{3}(b-2)^{3/5} - \frac{5}{3}(-2)^{3/5}\right] = \frac{5}{3}\,2^{3/5} \quad (4)$$

Q#2 (20 points)Check if the following integrals converges or diverges

a- $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

LCT with $\frac{1}{x^{3/2}}$

$\lim_{x\to\infty} \frac{\sqrt{x+1}}{x^2}\cdot\frac{x^{3/2}}{1} = \lim_{x\to\infty}\frac{\sqrt{x^4+x^{3/2}}}{x^2} = 1$ 3

since $\int_1^\infty \frac{1}{x^{3/2}}$ converges (P-test) 3

so by LCT, $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges 4

b- $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

LCT with $\frac{1}{\sqrt{x}}$

$\lim_{x\to\infty} \frac{1}{\sqrt{x-1}}\cdot\sqrt{x} = 1$ 3 and since $\int_2^\infty \frac{1}{\sqrt{x}}$ diverges 3

by P test so $\int_2^\infty \frac{dx}{\sqrt{x-1}}$ diverges. 4

or can be done by Evaluate the Integral

Q#3 (20 points) Find the limits of the following sequences:

1- $a_n = \frac{\ln(n+1)}{\sqrt{n}}$

$$\lim a_n = \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} = \frac{1}{\sqrt{n}} = 0$$

2- $a_n = \frac{(\cos n)^2}{2^n}$

$$0 \le \frac{(\cos n)^2}{2^n} \le \frac{1}{2^n}$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} \frac{(\cos n)^2}{2^n} \le \lim_{n\to\infty} \frac{1}{2^n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{(\cos n)^2}{2^n} = 0$$

3- $a_n = (\frac{3n+1}{3n-1})^n$

$$\lim a_n = \lim_{n\to\infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{3n}\right)^n}{\left(1-\frac{1}{3n}\right)^n} = \frac{e^{1/3}}{e^{-1/3}} = e^{2/3}$$

4- $a_n = \sqrt[n]{3^{2n+1}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(3^{2n+1}\right)^{1/n} = \lim \left(3^{2n}\cdot 3\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(3^2 \cdot 3^{1/n}\right) = 3^2 \cdot 1$$

$$= 3^2$$

5- $a_n = (3^n + 5^n)^{\frac{1}{n}}$

$$b_n = \ln(3^n+5^n)^{1/n}, \quad a_n = e^{b_n}, \quad \lim a_n = e^{\lim b_n}.$$

$$\lim b_n = \lim \frac{1}{n} \ln(3^n+5^n)$$

$$= \left(\lim \frac{\ln(3^n+5^n)}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{3^n+5^n} \cdot 3^n \ln 3 + 5^n \ln 5}{1}$$

$$= \lim_{n\to\infty} \frac{(\frac{3}{5})^n \ln 3 + \ln 5}{(\frac{3}{5})^n + 1} = \ln 5$$

$$\lim a_n = e^{\lim b_n} = e^{\ln 5} = 5$$

#4 (40 p0ints) Which of the following series converges or diverges, if the series converges find its sum (if possible)

1- $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

$S_1 = a_1 = 1 - \frac{1}{\sqrt{2}}$

$S_2 = a_1 + a_2 = (1 - \cancel{\frac{1}{\sqrt{2}}}) + (\cancel{\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{3}}) = 1 - \frac{1}{\sqrt{3}}$

$S_3 = a_1 + a_2 + a_3 = 1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} = 1 - \frac{1}{\sqrt{4}}$

$\vdots$

$S_n = 1 - \frac{1}{\sqrt{n+1}} \Rightarrow \lim_{n\to\infty} S_n = 1 = \sum\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

2- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$

$= 3(-1) \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$

$= -3\left(\frac{-\frac{1}{2}}{1+\frac{1}{2}}\right) = \frac{3/2}{3/2} = 1$

3- $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}$

nth term test the series diverges

4- $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ by Integral test

$\int_3^{\infty} \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = \frac{-1}{\ln x}$

$\lim_{b\to\infty}\left[\frac{-1}{\ln x}\Big|_3^b\right] = \lim_{b\to\infty}\left[\frac{-1}{\ln b} + \frac{1}{\ln 3}\right] = \frac{1}{\ln 3}$

so the series converges

5- $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

$n > \ln n \Rightarrow n > \ln n$

$\Rightarrow \ln n > \ln(\ln n) \Rightarrow n > \ln n > \ln(\ln n)$

$\Rightarrow \sum \frac{1}{n} < \sum \frac{1}{\ln n} < \sum \frac{1}{\ln(\ln n)}$

diverges p test $\Rightarrow \sum \frac{1}{\ln(\ln n)}$ diverges

6- $\sum_{n=1}^{\infty} \left(\frac{n+2^n}{n^2 2^n}\right)$

$= \sum \frac{n}{n^2 2^n} + \sum \frac{2^n}{n^2 2^n}$

$= \sum \frac{1}{n 2^n} + \sum \frac{1}{n^2}$

$< \sum \frac{1}{2^n} + \sum \frac{1}{n^2} \rightarrow$ converge p test, $p > 1$

geometric $\Rightarrow$ converges

7- $\sum_{n=1}^{\infty} \frac{(n+2)(n+1)}{n!}$

ratio

$\lim_{n\to\infty} \frac{(n+3)(\cancel{n+2})}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{(\cancel{n+2})(n+1)} = 0$

8- $\sum_{n=1}^{\infty} \left(\frac{(-2)^{n+1}}{n+5^n}\right)$

absolutely? $\sum \frac{2^{n+1}}{n+5^n} \leq \sum \frac{2^{n+1}}{5^n}$ (since $n+5^n > 5^n$)

$\qquad = \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$

Geometric converges

by D.C.T the series converges